ALL SAINTS' COLLEGE

Mathematics Specialist Test 1 2016

COMPLEX NUMBERS

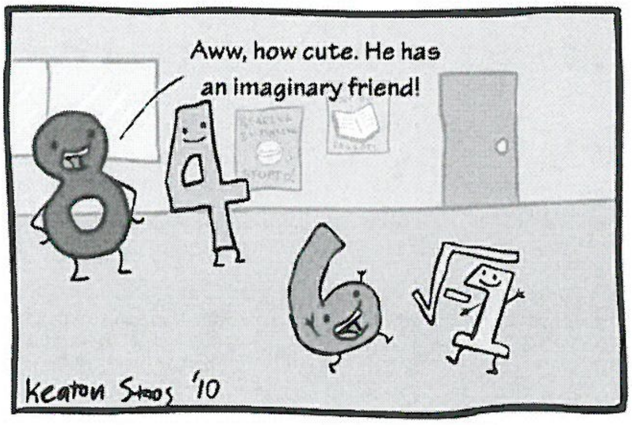
RESOURCE FREE

NAME: SOLUTIONS

TEACHER: MLA

28 marks

28 minutes



SCSA formulae sheets may be used in this section

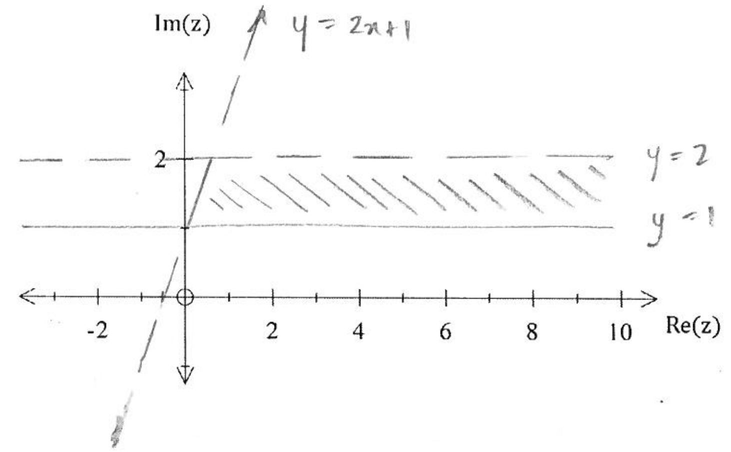
Question 1 [3, 2, 2 and 2 = 9 marks]

Represent the following regions on separate Argand diagrams:

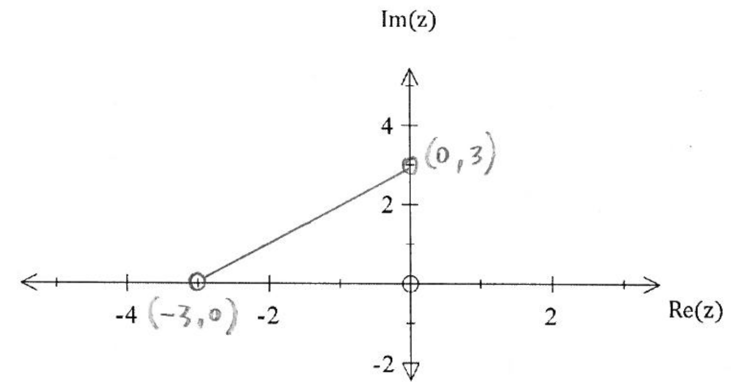
(a) $\text{Im}(z) \leq 2\text{Re}(z) + 1 \;\cap\; 1 \leq \text{Im}(z) < 2$

$y \leq 2x+1$

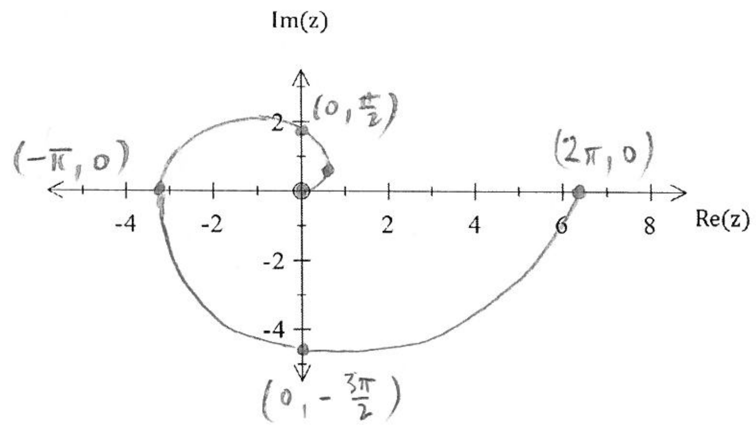
$1 \leq y < 2$



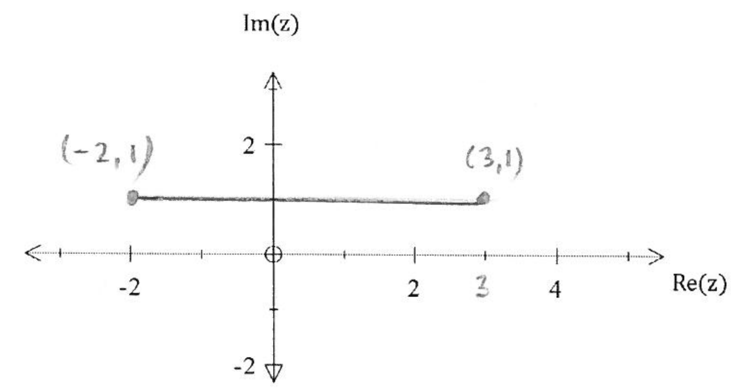
(b) $\arg(z - 3i) = \arg(z + 3) + \pi$



(c) $|z| = \arg(z)$



(d) $|(z - (3 + i)| + |z + 2 - i| = 5$



Question 2 [3, 2 & 3 = 8 marks]

(a) If $z = r\,\text{cis}(\alpha)$, prove that $z^{-1} = \frac{\bar{z}}{r^2}$

$$\text{LHS} = z^{-1} = [r\,\text{cis}(\alpha)]^{-1}$$
$$= \frac{1}{r}\,\text{cis}(-\alpha) \quad \text{... deMoivre's theorem}$$
$$= \frac{r\,\text{cis}(-\alpha)}{r^2}$$
$$= \frac{\bar{z}}{r^2}$$
$$= \text{RHS} \blacksquare$$

or $\text{LHS} = z^{-1} = \frac{1}{z}$
$$= \frac{1}{r\,\text{cis}(\alpha)}$$
$$= \frac{1}{r\,\text{cis}\,\alpha} \cdot \frac{r\,\text{cis}(-\alpha)}{r\,\text{cis}(-\alpha)}$$
$$= \frac{r\,\text{cis}(-\alpha)}{r^2}$$
$$= \frac{\bar{z}}{r^2}$$
$$= \text{RHS} \blacksquare$$

(b) Show that $\cos(\theta) - i\sin(\theta) = \text{cis}(-\theta)$

$$\text{LHS} = \cos(\theta) - i\sin(\theta)$$
$$= \cos(-\theta) + i\sin(-\theta)$$
$$= \text{cis}(-\theta)$$
$$= \text{RHS} \blacksquare$$

(c) Express $z + \bar{z} = (z)(\bar{z})$ in Cartesian form. Describe the locus of z.

Let $z = x + iy$

$$x + iy + x - iy = (x+iy)(x-iy)$$
$$2x = x^2 + y^2$$
$$x^2 - 2x + y^2 = 0$$
$$(x-1)^2 + y^2 = 1$$

$\Rightarrow$ circle, centre $(1, 0)$, radius 1 unit

Question 3 [3 & 2 = 5 marks]

(a) Use de Moivre's theorem to solve $z^3 = -8$, leaving answers in polar form.

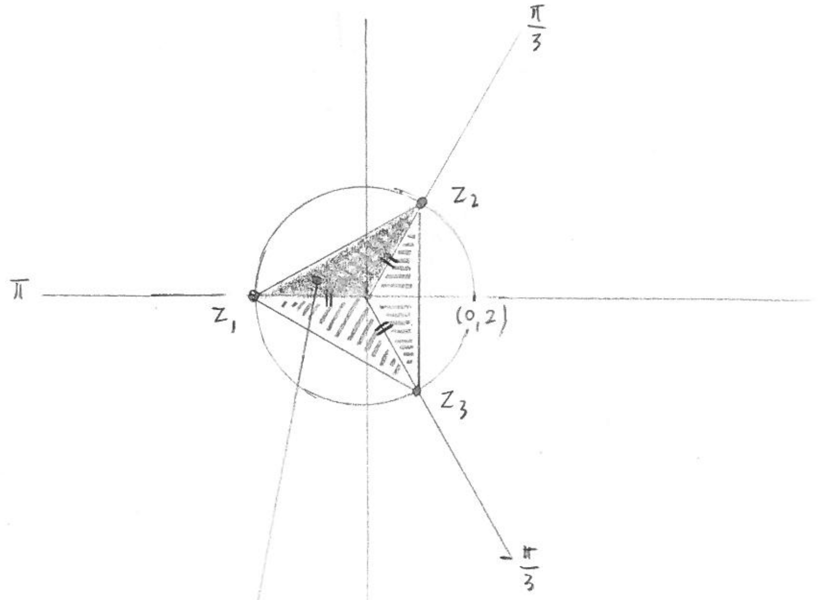
$$z^3 = 8 \operatorname{cis}(\pi + 2k\pi), \quad \forall k \in J$$

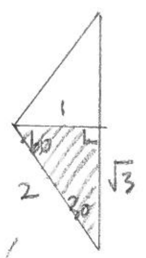
$$z = 2 \operatorname{cis}\left(\frac{\pi + 2k\pi}{3}\right)$$

when $k = -1$, $z_1 = 2 \operatorname{cis}(\pi)$

$k = 0$, $z_2 = 2 \operatorname{cis}\left(\frac{\pi}{3}\right)$

$k = 1$, $z_3 = 2 \operatorname{cis}\left(-\frac{\pi}{3}\right)$





(b) Determine the exact area of the polygon whose vertices are the solutions found above.

$$\text{Area}(\Delta) = \frac{1}{2} ab \sin C$$

$$= \frac{1}{2} \times 2 \times 2 \times \sin\left(\frac{2\pi}{3}\right)$$

$$= \sqrt{3} \text{ units}^2$$

3 such triangles $= 3\sqrt{3}$ units.

OR

$$\text{Area}(\Delta) = \frac{1}{2} \times b \times h$$

$$= \frac{1}{2} \times 1 \times \sqrt{3}$$

$$= \frac{\sqrt{3}}{2} \text{ units}^2$$

6 such triangles $= 3\sqrt{3}$ units^2

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

Question 4 [6 marks]

$\sin(2\theta) = \frac{1}{2i}\left(z^2 - \frac{1}{z^2}\right)$

Consider the identities $z^n + \frac{1}{z^n} = 2\cos(n\theta)$ and $z^n - \frac{1}{z^n} = 2i\sin(n\theta)$.

Use one or both of these identities to prove that $6\sin(2\theta) + 3\sin(4\theta) = 12\sin(2\theta)\cos^2(\theta)$.

$$\begin{aligned}
RHS &= 12\sin(2\theta)\cos^2(\theta) \\
&= 12\left[\frac{1}{2i}\left(z^2 - \frac{1}{z^2}\right)\right].\left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^2 \\
&= \frac{6}{i}\left(z^2 - \frac{1}{z^2}\right)\cdot\frac{1}{4}\left(z^2 + 2 + \frac{1}{z^2}\right) \\
&= \frac{3}{2i}\left(z^2 - \frac{1}{z^2}\right)\left(z^2 + 2 + \frac{1}{z^2}\right) \\
&= \frac{3}{2i}\left(z^4 + 2z^2 + 1 - 1 - \frac{2}{z^2} - \frac{1}{z^4}\right) \\
&= \frac{3}{2i}\left[\left(z^4 - \frac{1}{z^4}\right) + 2\left(z^2 - \frac{1}{z^2}\right)\right] \\
&= \frac{3}{2i}\left(z^4 - \frac{1}{z^4}\right) + \frac{3}{i}\left(z^2 - \frac{1}{z^2}\right) \\
&= 3\sin(4\theta) + 6\sin(2\theta) \\
&= LHS \quad \blacksquare
\end{aligned}$$

End of Section 1

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COMPLEX NUMBERS

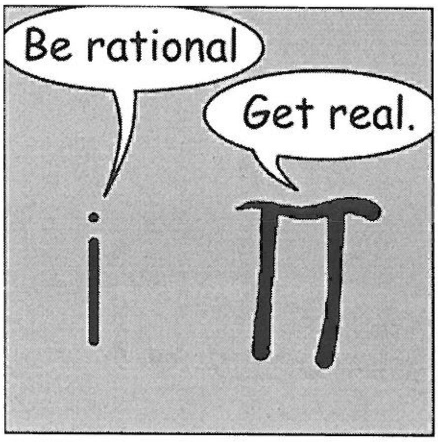
RESOURCE RICH

NAME: Solutions

TEACHER: MLA

22 marks

22 minutes



SCSA formulae sheets, ClassPads and a single A4 sheet of notes may be used in this section

Clear working must be shown in order to be awarded full marks

Question 5 [3 & 3 = 6 marks]

(a) The polynomial $2x^3 + bx^2 + c$ has a factor $(x + 1)$ and leaves a remainder of 16 when it is divided by $(x - 3)$. Find the values of b and c.

$f(-1) = -2 + b + c$
$= 0$
$\Rightarrow b + c = 2$ ①

$f(3) = 54 + 9b + c$
$= 16$
$\Rightarrow 38 + 9b + c = 0$ ②

ClassPad : $b = -5, \ c = 7$

(b) If $(x - a)^2$ is a factor of the real polynomial $f(x)$, then $(x - a)$ is a factor of $f'(x)$, where $f'(x)$ is the derivative of $f(x)$ with respect to x.

Knowing this, if $(x + 2)^2$ is a factor of $2x^4 + bx^3 + cx^2 - 4$, determine the values of b and c.

$f(-2) = 32 - 8b + 4c - 4$
$= 0$
$\Rightarrow 8b - 4c - 28 = 0$ ①

$f'(x) = 8x^3 + 3bx^2 + 2cx$

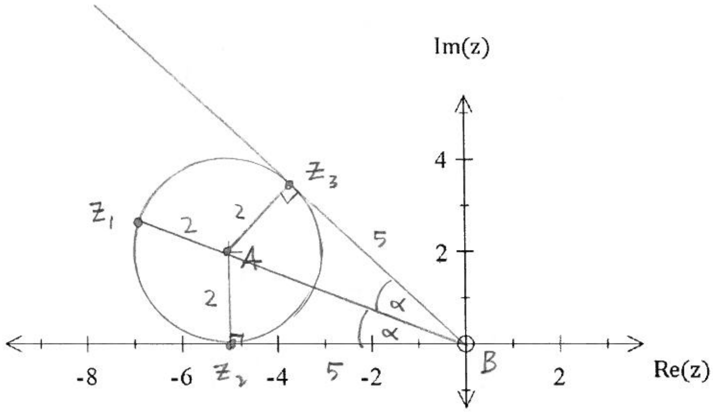
$f'(-2) = -64 + 12b - 4c$
$= 0$
$\Rightarrow 4c - 12b + 64 = 0$ ②

ClassPad : $b = 9, \ c = 11$

Question 6 [2, 1, 2 = 5 marks}

For $\{z: |z + 5 - 2i| = 2\}$, determine:

(a) The exact maximum possible value of $|z|$



$$\overline{AB} = \sqrt{2^2 + 5^2}$$
$$= \sqrt{29} \quad \Rightarrow \quad |z_1| = 2 + \sqrt{29} \text{ units.}$$

(b) The maximum possible value of $\arg(z)$

$$\arg(z_2) = \pi \text{ radians or } 180°$$

(c) The minimum possible value of $\arg(z)$, correct to 1 decimal place.

$$\arg(z_3) = 180 - 2\alpha$$
$$= 180 - 2\arctan\left(\frac{2}{5}\right)$$
$$= 136.4°$$

Question 7 [4 & 1 = 5 marks]

(a) Determine the Cartesian equation represented by $\{z: |z-(10+5i)| = 3\,|z-(2-3i)|\}$

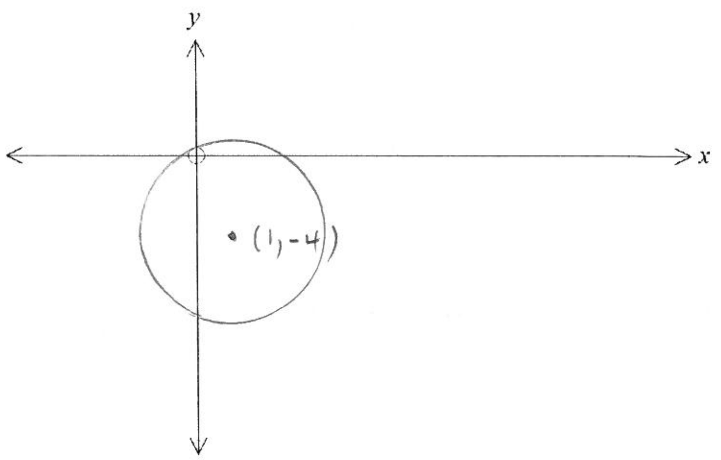
let $z = x + iy$

$$|x + iy - 10 - 5i| = 3|x + iy - 2 + 3i|$$
$$|(x-10) + (y-5)| = 3\,|(x-2) + (y+3)|$$
$$(x-10)^2 + (y-5)^2 = 9\left[(x-2)^2 + (y+3)^2\right]$$
$$x^2 - 20x + 100 + y^2 - 10y + 25 = 9(x^2 - 4x + 4 + y^2 + 6y + 9)$$
$$x^2 - 20x + 100 + y^2 - 10y + 25 = 9x^2 - 36x + 36 + 9y^2 + 54y + 81$$
$$8 = 8x^2 - 16x + 8y^2 + 64y$$
$$1 = x^2 - 2x + y^2 + 8y$$
$$1 = (x-1)^2 - 1 + (y+4)^2 - 16$$
$$\therefore\ 18 = (x-1)^2 + (y+4)^2$$

$\rightarrow$ circle, centre $(1,-4)$, radius $3\sqrt{2}$

($\doteqdot 4.24$ units)

(b) Sketch the locus defined in (a)



Question 8 [6 marks]

Solve $z^3 + (1+i)z^2 + (2+i)z + 2 = 0, \forall z \in C$, leaving answers in exact form.

$$f(1) = 1 + 1 + i + 2 + i + 2$$
$$\neq 0$$

$$f(-1) = -1 + 1 + i - 2 - i + 2$$
$$= 0$$

$\therefore$ $(z+1)$ is a factor of $f(z)$

$$\Rightarrow \quad (z+1)(az^2 + bz + c) = z^3 + (1+i)z^2 + (2+i)z + 2$$

By Inspection : $az^3 = z^3$ $\quad$ $c = 2$ $\quad$ $cz + bz = (2+i)z$

$a = 1$ $\quad$ $(b+c)z = (2+i)z$

$b + c = 2 + i$

$b = i$

That is, $(z+1)(z^2 + iz + 2) = 0$

$(z+1)(z+2i)(z-i) = 0$

$\therefore$ $z = \underline{\underline{-1, -2i \text{ or } i}}$.

Quadratic Formula :

$$z = \frac{-i \pm \sqrt{-1 - 4(1)(2)}}{2}$$

$$= \frac{-i \pm \sqrt{-9}}{2}$$

$$= \frac{-i \pm 3i}{2}$$

$$= -2i \text{ or } i$$

note.

$$\begin{array}{ccc} z & & 2i \\ & \times & \\ z & & -i \end{array}$$

$\Rightarrow$ factors

$(z+2i)(z-i)$